Solution to Waveguide Problems by Successive Extrapolated Relaxation

E. DELLA TORRE AND W. KINSNER

***Abstract*—A successive extrapolated relaxation (SER) technique has been developed to solve elliptic partial difference equations iteratively. SER is more efficient than optimized successive over-relaxation (SOR) and permits useful solutions of waveguide modes using finite difference methods.**

Introduction

Beaubien and Wexler [1], [2] have discussed the solution of waveguide problems by finite difference methods. The resulting method PDSOR (positive definite successive over-relaxation) uses a one-dimensional search technique to obtain the best value of the over-relaxation factor. Such a technique suffers from suboptimal choice of the over-relaxation factor during the final steps of the solution. They cannot use the optimized SOR (successive over-relaxation) [3] because their matrix C, although positive definite, does not possess Young's property A [4].

A new method called SER (successive extrapolated relaxation) [5] has been developed to solve elliptic partial difference equations. It has been shown [5] that SER is at least as efficient as the optimized SOR. Since SER does not require that the system matrix possess Young's property A, it may be applied directly to the Beaubien and Wexler formulation with a resulting increase in speed. If the problem can be reformulated so that the system matrix does possess Young's property A, then a refinement of SER called SEOR (successive extrapolated optimized relaxation) may be used which optimizes the pertinent parameter.

The SER Method

Let us define $V_{k,l}^{(n)}$ to be the value of the potential at the k, l lattice point after the nth iteration. During the iterative solution, the sequence $V_{k,l}^{(n-2)}$, $V_{k,l}^{(n-1)}$, $V_{k,l}^{(n)}$ obtained by any successive relaxation process may be plotted as shown in Fig. 1. If one assumes that the approach to the asymptotic value is characterized by an exponential behavior

$$V_{k,l}^{(n)} = V_{k,l}^{(\infty)} + (V_{k,l}^{(0)} - V_{k,l}^{(\infty)})\alpha^n \tag{1}$$

then an approximation $A_{k,l}^{(n)}$ to the asymptotic value is given by Aitken's formula

$$A_{k,l}^{(n)} = \frac{[V_{k,l}^{(n-1)}]^2 - V_{k,l}^{(n-2)}V_{k,l}^{(n)}}{2V_{k,l}^{(n-1)} - V_{k,l}^{(n-2)} - V_{k,l}^{(n)}}. \tag{2}$$

If the sequence converges geometrically, then (2) is the solution to the problem. If the convergence is quasi-geometric, then (2) is much closer to the final answer than $V_{k,l}^{(n)}$. In general, the convergence is linear, and (2) has to be modified in order to assure convergence.

The first modification involves bounding the extrapolation as illustrated in Fig. 2. The second modification is due to the fact that (2) extrapolates the wrong way if there is an apparent divergence in the sequence $\{V_{k,l}^{(i)}\}$. This is solved by reflecting the $n-2$ point as shown in Fig. 3. The resulting extrapolation formula is given by

$$E_{k,l}^{(n+1)} = \begin{cases} V_{k,l}^{(n)} + 2(1+\gamma)\Delta_2, & \Delta_1\Delta_2 \geq 0 \quad |\Delta_2| \leq |\Delta_1| \\ V_{k,l}^{(n)} + (3+2\gamma)\Delta_1, & \Delta_1\Delta_2 \geq 0 \quad |\Delta_2| > |\Delta_1| \\ V_{k,l}^{(n)}, & \Delta_1\Delta_2 < 0 \end{cases} \tag{3}$$

where

$$\Delta_1 = V_{k,l}^{(n-1)} - V_{k,l}^{(n-2)} \tag{4}$$

$$\Delta_2 = V_{k,l}^{(n)} - V_{k,l}^{(n-1)} \tag{5}$$

and γ is a constant of the order of $\frac{1}{2}$. In SEOR, γ is optimized.

Illustrative Examples

The dominant mode for the L-shaped region consisting of three unit squares was solved using SER. It took 100 iterations (12.3 s) as compared to 300 iterations (29.5 s) when using the Gauss–Seidel method. The mesh size was $h = 1/50$ for $a = 1$, and the error criterion was the residual ratio of 10^{-4}. The five-point operator was used in the computational process. The dense mesh size used in this solution caused the irregularities of the field distribution introduced by the re-entrant corner of the guide to be negligible [6].

Higher order modes for circular waveguide were solved using Wexler's PDSOR program [7]. The extrapolation technique was introduced into the program and reduced the computation time by the factor of two. The results were consistent with those obtained by the program without extrapolation.

Manuscript received May 30, 1972; revised Feburary 15, 1973. This work was supported by the National Research Council of Canada.

The authors are with the Communications Research Laboratory, Department of Electrical Engineering, McMaster University, Hamilton, Ont., Canada.

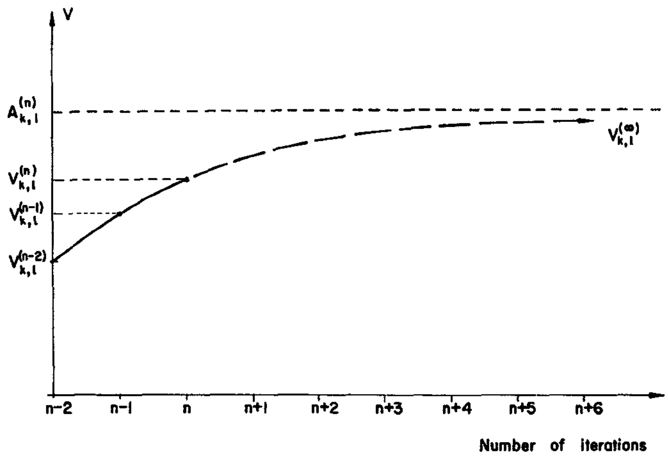

Fig. 1. The sequence $V_{k,l}^{(n)}$ obtained by any successive relaxation process indicating the implied asymptote.

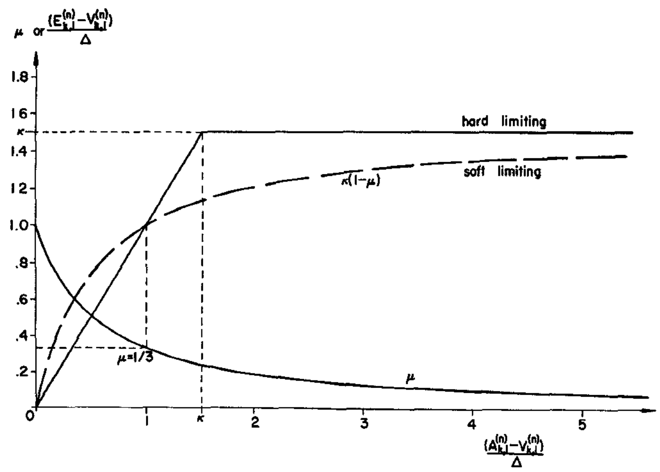

Fig. 2. Bounding the extrapolation process.

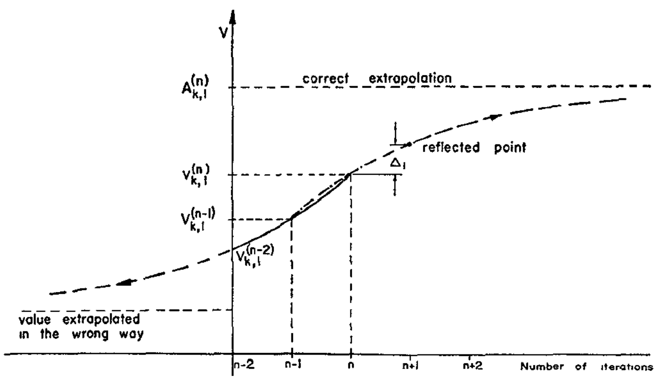

Fig. 3. Reflection of the $(n-2)$ point in order to obtain correct extrapolation.

Since SEOR requires the system matrix to possess Young's property A, one has to use a seventeen-point operator [8] in conjunction with the five-point Laplacian instead of the thirteen-point operator as used in [7].

References

[1] M. J. Beaubien and A. Wexler, "Iterative, finite difference solution of interior eigenvalues and eigenfunctions of Laplace's operator," *Computer J.*, vol. 14, pp. 263–269, Mar. 1971.
[2] ——, "Unequal-arm finite-difference operators in the positive-definite successive over-relaxation (PDSOR) algorithm," *IEEE Trans. Microwave Theory Tech. (1970 Symposium Issue)*, vol. MTT-18; pp. 1132–1149. Dec. 1970.
[3] B. A. Carré, "The determination of the optimum accelerating factor for successive over-relaxation," *Computer J.*, vol. 4, pp. 73–78, 1961.
[4] D. M. Young, "Generalizations of property A and consistent ordering," Center for Numerical Analysis, Univ. of Texas, Austin, Rep. CNA-6, 1970.
[5] E. Della Torre and W. Kinsner, "Convergence of the successive extrapolated relaxation (SER) method," *J. Inst. Math. Its Appl.*, to be published.
[6] J. B. Davies and P. Nagenthiram, "Irregular fields, nonconvex shapes and the point-matching method for hollow waveguides," *Electron. Lett.*, vol. 7, no. 14, pp. 401–404, 1971.
[7] M. Beaubien and A. Wexler, "Higher waveguide modes by positive definite SOR," *IEEE Trans. Microwave Theory Tech.* (Compt. Program Descr.), vol. MTT-19, pp. 839–840, Oct. 1971.
[8] G. J. Tee, "A novel finite-difference approximation to the biharmonic operator," *Computer J.*, vol. 6, pp. 177–192, 1963.

Highly Stabilized IMPATT Oscillators at Millimeter Wavelengths

S. NAGANO AND S. OHNAKA

***Abstract*—Highly stabilized IMPATT oscillators at millimeter wavelengths have been developed. The IMPATT diode is mounted in the coaxial-waveguide circuit at the detuned open position, and is series-resonant at the design frequency. The frequency stability and power output of $\pm 5\times10^{-5}/\mp 20\,°C$ and 50 mW, respectively, have been obtained at 80 GHz.**

Highly stabilized solid-state oscillators at millimeter wavelengths are presently required, especially for the realization of a phase-shift keyed (PSK) guided millimeter-wave transmission system. Several types of oscillators have been developed, mainly in the microwave region [1]–[3], but also in the millimeter-wave region [4]. This short paper describes the design considerations and the experimental results of the reaction-cavity controlled IMPATT oscillator having good frequency stability in the millimeter-wave region.

The construction of the oscillator is essentially the same as the highly stabilized *Ka*-band Gunn oscillator [5], but the operating point is made more suitable for the stabilization. In the millimeter-wave region it is difficult to obtain superior frequency stability, because the unloaded Q of the cavity is much smaller than that of the cavity in the microwave region (for instance, Q_0 at 100 GHz is about one-third of Q_0 at 10 GHz, even if the cavity is ideally constructed). This problem can be solved by utilizing a higher order resonant mode of the cavity and also making the oscillator operate at the point where the increment of the susceptance against frequency is maximum. In order to realize the latter, we have placed the diode at the detuned open position of the cavity and also made the diode series-resonant at the design frequency.

The equivalent circuit of the oscillator is shown in Fig. 1. The assumptions are made that 1) the electronic susceptance of the device is negligibly smaller than that due to the junction capacity C_j, 2) the diode has only a series lumped inductance L_e, except for the negative conductance of the device G_d and the capacitance C_j, 3) the diode is connected to the transmission line of characteristic impedance Z_2 at the detuned open position of the reaction cavity $(1-1'$ in Fig. 1), and 4) the diode is also connected to the line of characteristic impedance Z_0 through an ideal transformer of turns ratio $n:1$. The locus of the admittance Y_c, which the negative conductance sees, is shown in Fig. 2. The assumptions are made that the series-

Manuscript received May 30, 1972; revised February 23, 1973.
The authors are with the Central Research Laboratories, Nippon Electric Company, Ltd., Kawasaki, Japan.

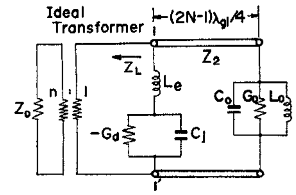

Fig. 1. Equivalent circuit of stabilized IMPATT oscillator.

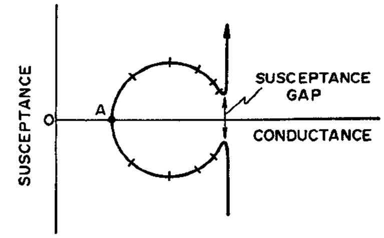

Fig. 2. Admittance locus of reaction-cavity controlled oscillator.

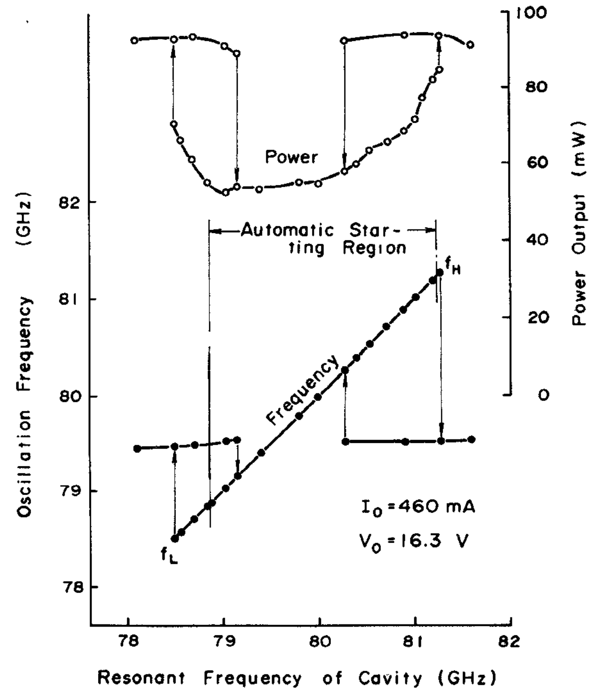

Fig. 3. Tuning characteristic of an 80-GHz IMPATT oscillator.

resonant frequency of the diode $\omega_s(=1/\sqrt{L_eC_j})$ is equal to the resonant frequency of the cavity $\omega_0(=1/\sqrt{L_oC_o})$, and $Q_e(=\omega_0C_0Z_2) \gg Q_1(=\omega_sL_e/Z_2)$. The arrow represents the direction of the increment of the frequency, and the cross lines equal increments of frequency. The oscillator can be stably operated at point A, where the increment of the susceptance against frequency is maximum. As the admittance locus has a "susceptance gap" (refer to Fig. 2), the single-mode operation can be obtained in a narrow tuning range.

An 80-GHz IMPATT oscillator has been designed, based on the above design considerations. The reaction cavity is made of "super-Invar," whose resonant mode is the cylindrical TE_{013} mode. The measured value of Q_0 was 11 000 (cf. theoretical value is about 16 000). In order to make the diode series-resonant at 80 GHz, a coaxial-waveguide circuit [6] has been adopted, where the dimensions of the waveguide cross section are 3.1×1.0 mm. The distance between the diode and the detuned short position of the cavity is $3\,\lambda_g/4$ at 80 GHz.

Fig. 3 shows the tuning characteristic of the oscillator. In the frequency range from 79.2 to 80.2 GHz, the oscillator operates only